Proc. ISSAC '98, Rostock, Germany, 1998, pp. 284-289

High degree solutions of low degree equations

Jürgen Gerhard

Fachbereich 17 Mathematik-Informatik
Universität-GH Paderborn
D-33095 Paderborn, Germany
e-mail: jngerhar@uni-paderborn.de
www: http://www-math.uni-paderborn.de/~jngerhar/

Extended Abstract, August 21, 1998

Abstract

We exhibit a class of proper hypergeometric expressions which lead to a key equation with coefficients of degree at most two and a unique solution of arbitrarily high degree in Gosper's algorithm from 1978 for indefinite hypergeometric summation. We investigate similar classes for the related problems of indefinite integration and q-hypergeometric summation.

1 Introduction

Symbolic summation is the task where an expression $g(k)$, depending on an integral parameter k, is given and we want to find another expression $f(k)$ of the same kind such that

$$f(n) = \sum_{0\leq k<n} g(k) \text{ for } n \in \mathbb{N}. \qquad (1)$$

A central role is played by the *shift operator* E with $(Ef)(k) = f(k+1)$ and the *difference operator* $\Delta = E - I$ with $(\Delta f)(k) = f(k+1) - f(k)$, since $\Delta f = g$ and $f(0) = 0$ imply that f satisfies (1). Familiar examples are sums like

$$\frac{n^2-n}{2} = \sum_{0\leq k<n} k \text{ and } \frac{q^n-1}{q-1} = \sum_{0\leq k<n} q^k.$$

For the rich class of *hypergeometric expressions*, which include products of rational functions, factorials, binomial coefficients, and exponentials, the summation problem has first been solved by Gosper [13]. Other contributions are due to Abramov [1], Moenck [18], Karr [15, 16], Zeilberger [29, 28, 30], Wilf & Zeilberger [26, 27], Lisoněk *et al.* [17], Pirastu & Strehl [22], and Paule [19]; Paule & Strehl [20] give a unifying overview of more recent works. The method of Wilf and Zeilberger, which uses a variant of Gosper's algorithm as a subroutine, is able to produce routinely short proofs of all kinds of combinatorial identities, among them such famous ones as Dixon's theorem [10] and the first Rogers-Ramanujan identity [11]. The related problem of solving difference and differential equations with rational coefficients is investigated by Risch [23], Rothstein [24], Davenport [9], Abramov [2, 3, 4], Bronstein [6, 7], Petkovšek [21], and Abramov *et al.* [5].

We report on a particular class of solutions of the *key equation* in Gosper's algorithm, where polynomials $r, s, v \in F[x]$ over a field F of characteristic zero are given and a polynomial $u \in F[x]$ satisfying

$$r \cdot Eu - s \cdot u = v$$

is sought. Gosper shows that there are two essentially different possibilities for the degree of such a solution u: it can be expressed either solely in terms of the degrees of r, s, v, or solely in terms of certain *coefficients* of r and s. Up to now, there seem to be known only few nontrival examples (where r, s, v result from a proper (i.e. nonrational) hypergeometric summation problem) leading to the second case; Lisoněk *et al.* [17] give some. Since the degree of u affects the running time of Gosper's algorithm, it is interesting to know whether the second case is of real impact or merely a curiosity.

The main contribution of this paper is that we exhibit a certain class of proper hypergeometric summation problems leading to a key equation with degrees of r, s, v at most two and a unique solution u of arbitrarily high degree (Theorem 2). We also consider similar classes for the differential case (Theorem 6) and the q-difference case (Theorem 9), and give examples in all three cases. The paper is organized as follows. In Section 2, we introduce the problem and prove the main result. Sections 3 and 4 discuss the differential and the q-difference analog, respectively. We conclude in Section 5 with a preview to future work.

2 The difference case

Let F be a field of characteristic zero and $K = F(x, g)$ an extension of the field $F(x)$ of rational functions. We assume that K is a difference field extension of $F(x)$, with an automorphism $E: K \longrightarrow K$ such that E is the shift operator $E: \rho \longmapsto \rho(x+1)$ on $F(x)$, and that $Eg = \sigma g$ for some $\sigma \in F(x)$, i.e., g is a *hypergeometric expression*. For example, the Γ function is hypergeometric, and $(E\Gamma)(x) = \Gamma(x+1) = x\Gamma(x)$ shows that we have $\sigma = x$. Any rational function is hypergeometric.

Gosper's [13] algorithm solves the *hypergeometric summation problem*: Is there another hypergeometric expression f such that $\Delta f = Ef - f = g$, and if so, then compute one. He has also shown that if such an f exists, then $f = \tau g$ for some $\tau \in F(x)$, i.e., f is a rational multiple of g and already belongs to K. From

$$g = \Delta f = \Delta(\tau g) = E(\tau g) - \tau g = (E\tau \cdot \sigma - \tau)g,$$

we obtain the rational difference equation

$$\sigma \cdot E\tau - \tau = 1 \qquad (2)$$

for the rational function τ.

Gosper's algorithm solves (2) by finding polynomials $r, s, v \in F[x]$ such that

$$\sigma = \frac{r}{Es} \cdot \frac{Ev}{v}, \quad \gcd(r, E^i s) = 1 \text{ for } i \in \mathbb{N}_{>0}. \qquad (3)$$

If we stipulate that v and s be monic and $\gcd(r, v) = \gcd(s, v) = 1$, then r, s, v exist and are uniquely determined by σ, and (3) is called the *Gosper-Petkovšek representation* of σ (see Petkovšek [21]).

The final step in Gosper's algorithm is to solve the *key equation*

$$r \cdot Eu - s \cdot u = v \qquad (4)$$

for a polynomial $u \in F[x]$. If such a polynomial exists, then $\tau = us/v$ solves the rational equation (2), and $f = \tau g$ is a solution of the hypergeometric summation problem $\Delta f = g$. If (4) is unsolvable, then the latter has provably no solution.

Solving the key equation is usually done by determining a degree bound e on u and the method of undetermined coefficients, where u of degree e with unknown coefficients is plugged into (4) and then the linear system equivalent to (4) is solved for the coefficients of u (see von zur Gathen & Gerhard [12] for a different approach). The degree bound is provided by the following lemma, also due to Gosper [13].

Lemma 1 *Let $r = r_m x^m + r_{m-1}x^{m-1} + \cdots + r_0$, $s = x^n + s_{n-1}x^{n-1} + \cdots + s_0$, v and $u = u_e x^e + \cdots + u_0$ be nonzero polynomials in $F[x]$, with nonzero $r_m, u_e \in F$ (where we set $r_{m-1} = 0$ if $m = 0$, and similarly $s_{n-1} = 0$ if $n = 0$), such that (4) holds. Furthermore, let $\delta = s_{m-1} - r_{m-1} \in F$. Then*

(i) $\deg u = \deg v - \deg(r - s)$ *if* $\deg(r - s) \geq \deg(r + s)$,

(ii) $\deg u = 1 + \deg v - \deg r$ *if* $\deg(r - s) < \deg(r + s)$ *and* $\delta \notin \mathbb{N}$,

(iii) $\deg u \in \{1 + \deg v - \deg r, \delta\}$ *if* $\deg(r - s) < \deg(r + s)$ *and* $\delta \in \mathbb{N}$.

(iv) In the first two cases, (4) has at most one polynomial solution $u \in F[x]$. In the last case, we have $\delta \neq 1 + \deg v - \deg r$.

Lisoněk *et al.* [17] (where our "δ" is called "K_0") have shown that the key equation has at most one solution in every case when g is *proper* hypergeometric (i.e., hypergeometric but not rational). The case where $\deg u = \delta$ in the above lemma is surprising at first sight, since then the degree of the solution u to (4) does not depend on the degree of r, s, v but only on the coefficients of r and s. For the case where g is rational, Lisoněk *et al.* have shown that the solution space of (4) is either one-dimensional or empty. In the former case, there is a unique solution u of degree $1 + \deg v - \deg r$, and all other solutions are of the form $u + cu^*$ for a constant $c \in F$, where u^* is a nonzero solution of degree δ of the corresponding homogeneous difference equation $r \cdot Eu^* - s \cdot u^* = 0$, or equivalently,

$$\frac{s}{r} = \frac{Eu^*}{u^*}, \qquad (5)$$

which then is the Gosper-Petkovšek representation of s/r. (In fact, a more general form of the Gosper-Petkovšek representation is

$$\sigma = \frac{r}{s} \cdot \frac{u}{Eu} \cdot \frac{Ev}{v}, \quad \gcd(r, E^i s) = 1 \text{ for } i \in \mathbb{Z},$$

with polynomials $r, s, u, v \in F[x]$, which is unique if we require that s, u, v be monic and $\gcd(r, Eu) = \gcd(r, v) = \gcd(s, u) = \gcd(s, Ev) = \gcd(u, v) = 1$, and then g is rational if and only if $r = s = 1$.) Lisoněk *et al.* conclude that for rational g, the case $\deg u = \delta$ need not be considered.

If g is not rational, then at first sight it is not clear whether the key equation admits solutions of degree δ at all. Examples for this situation have been given by Lisoněk *et al.* [17]. We will now exhibit a class of hypergeometric expressions for which we show that (4) always has a unique solution of degree δ, which may be arbitrary large.

Let

$$\sigma = \frac{a}{b} = \frac{x^2 + a_1 x + a_0}{x^2 + (a_1 + d)x + b_0}, \qquad (6)$$

with $a_1, a_0, b_0 \in F$ and $d \in \mathbb{N}$ such that $d \geq 2$. We say that a and b are *shift-coprime* if $\gcd(a, E^i b) = 1$ for all $i \in \mathbb{Z}$, or equivalently, the resultant $R = \mathrm{res}_x(a(x), b(x + y)) \in F[y]$ has no integer zeroes. When $a_1 = a_0 = 0$, for example, then $R = (y^2 + dy + b_0)^2$, and a and b are shift-coprime if and only if $b_0 \neq i(d - i)$ for all integers i.

Theorem 2 *If $a = x^2 + a_1 x + a_0$ and $b = x^2 + (a_1 + d)x + b_0$ in $F[x]$, with $a_1, a_0, b_0 \in F$ and $d \in \mathbb{N}_{\geq 2}$, are shift-coprime, then we have $v = 1$ in (3), and there is a unique polynomial $u \in F[x]$ of degree $d - 2$ solving (4).*

Proof. Substituting $x - a_1/2$ for x is an automorphism of $F(x)$ and does not change our arguments in the sequel, and we may assume that $a_1 = 0$. The fact that a and b are shift-coprime implies that (6) is already in Gosper-Petkovšek form, i.e., we have $r = a$, $s = E^{-1}b = x^2 + (d-2)x + 1 - d + b_0$, and $v = 1$ in (3). Now

$$\deg(r - s) = \deg((2 - d)x + a_0 - b_0 + d - 1) < 2 = \deg(r + s),$$

$1 + \deg v - \deg r = -1$, and $\delta = d - 2$ in Lemma 1, which implies that $\deg u = \delta = d - 2 \geq 0$ if a solution u of the key equation (4) exists. (If $d < 2$ or $d \notin \mathbb{N}$, then Lemma 1 implies that (4) is unsolvable.)

The first order difference operator $L = rE - s$, which maps a polynomial $u \in F[x]$ to $Lu = r \cdot Eu - s \cdot u$, is linear in the coefficients of u. Thus (4) reads $Lu = 1$. If $u = u_{d-2}x^{d-2} + \cdots + u_1 x + u_0 \in F[x]$, then the coefficients of x^d and x^{d-1} in $r \cdot Eu - s \cdot u$ cancel, and hence L maps the $d - 1$-dimensional vector space $P_{d-1} \subseteq F[x]$ of all polynomials of degree less than $d - 1$ to itself. Since a and b are shift-coprime, so are r and s, whence (5) is unsolvable. By the results of Lisoněk *et al.* [17] discussed above, L is injective, and hence the restriction L_{d-1} of L to P_{d-1} is an automorphism of the vector space P_{d-1}. This implies that the key equation always has a unique solution u, and Lemma 1 implies that $\deg u = \delta = d - 2$. □

If we regard d as an indeterminate, then we are in the second case of Lemma 1, and $1+\deg v-\deg r=-1$ implies that (4) is unsolvable.

We note that the proof of Theorem 2 hinges upon the fact that $\deg a=\deg b=2$. If $a=x$ and $b=x+d$, then a and b are not shift-coprime, and if $a=x^k+a_{k-2}x^{k-2}+\cdots$ and $b=x^k+dx^{k-1}+b_{k-2}x^{k-2}+\cdots$ for some $k>2$, then we can only show that the linear operator L maps polynomials of degree less than $d-1$ to polynomials of degree less than $d+k-3>d-1$.

Example 3 *Let $t\in\mathbb{N}_{\geq 2}$, and define the hypergeometric expression g by*

$$g(k)=2^{4k}\binom{k+t}{k}^{-2}\binom{2k+2t}{k+t}^{-2}$$

for all $k\in\mathbb{N}$. Its term ratio is

$$\sigma(k) \;=\; \frac{g(k+1)}{g(k)}=\frac{(k+1)^2}{\left(k+\dfrac{2t+1}{2}\right)^2}.$$

If we let $d=2t+1$, then

$$\sigma=\frac{a}{b}=\frac{(x+1)^2}{\left(x+\dfrac{d}{2}\right)^2}=\frac{x^2+2x+1}{x^2+dx+\dfrac{d^2}{4}},$$

such that a and b are shift-coprime since d is odd. Theorem 2 implies that there is a unique polynomial $u\in F[x]$ of degree $d-4=2t-3$ solving (4) for $r=a$, $s=E^{-1}b$, and $v=1$, and thus the hypergeometric expression $f=u\cdot E^{-1}b\cdot g$ satisfies $\Delta f=g$ and $\sum_{0\leq k<n}g(k)=f(n)-f(0)$ for all $n\in\mathbb{N}$.

For example, when $t=6$ we obtain

$$\begin{aligned}\Delta\Bigg(&\frac{(4x^2+44x+121)(4x+7)}{281302875}(8388608x^8+117440512x^7\\&+658767872x^6+1881800704x^5+2862755840x^4\\&+2179846144x^3+648167040x^2+504000x\\&-496125)g\Bigg)=g,\end{aligned}$$

Example 4 *This is a variant of an example from Lisoněk* et al. *[17]. We let*

$$g(k)=\frac{\left((-5/2)^{\overline{k+1}}\right)^2}{(-1/3)^{\overline{k+1}}(-2/3)^{\overline{k+1}}}$$

for all $k\in\mathbb{N}$, where $c^{\overline{k+1}}=c(c+1)\cdots(c+k)$ for $c\in\mathbb{R}$. The term ratio is

$$\sigma=\frac{a}{b}=\frac{\left(x-\dfrac{5}{2}\right)^2}{\left(x-\dfrac{1}{3}\right)\left(x-\dfrac{2}{3}\right)}=\frac{x^2-5x+\dfrac{25}{4}}{x^2-x+\dfrac{2}{9}},$$

and a and b are shift-coprime. Theorem 2 with $d=4$ says that (4) has a unique polynomial solution $u\in F[x]$ of degree $d-2=2$ for $r=a$, $s=E^{-1}b$, and $v=1$. In fact, we have $u=4(2592x^2-12888x+15985)/11025$ and

$$\Delta\left(\frac{4}{11025}(2592x^2-12888x+15985)\left(x-\frac{4}{3}\right)\left(x-\frac{5}{3}\right)g\right)=g.$$

3 The differential case

Let $K=F(x,g)$ be an extension field of $F(x)$. If D is a derivative on K such that D is trivial on F, $D(x)=1$, and $Dg=\sigma g$ for a rational function $\sigma\in F(x)$, then g is a *hyperexponential expression.* The analog of the hypergeometric summation problem is then, given a hyperexponential expression g, to find another hyperexponential expression f such that $Df=g$; we call f an *integral* of g. For example, the expression $g=x\cdot\exp(x^2)$ is hyperexponential, $Dg=(1+2x^2)\exp(x^2)=(1+2x^2)/x\cdot g$, and an integral of g is $f=\frac{1}{2}\exp(x^2)=g/2x$. Any rational function is hyperexponential.

As in the difference case, it turns out that such an integral f—if it exists—is of the form $f=\tau g$ for a rational function $\tau\in F(x)$, and we obtain from

$$g=Df=D(\tau\cdot g)=D\tau\cdot g+\tau\cdot\sigma g$$

the *Risch differential equation*

$$D\tau+\sigma\tau=1 \tag{7}$$

for the rational function τ.

The solution of (7) can be reduced to finding polynomials $r,s,v\in F[x]$ such that r and v are monic,

$$\sigma=\frac{s}{r}+\frac{Dv}{v},\quad \gcd(r,s-i\cdot Dr)=1 \text{ for } i\in\mathbb{N} \tag{8}$$

(see, e.g., Bronstein [8], §6.1). If we furthermore require that r and v be coprime, then the decomposition (8) exists uniquely. (In fact, the more general decomposition

$$\sigma=D\left(\frac{c}{d}\right)+c_1\frac{Dv_1}{v_1}+\cdots+c_t\frac{Dv_t}{v_t}$$

from the integration algorithm for rational functions ([14], [24], [25]) is a refinement of (8). If $\sigma=a/b$ for coprime polynomials $a,b\in F[x]$ and b^* is the squarefree part of b, then the c_i are the distinct zeroes of the resultant $\mathrm{res}_x(b^*,a-y\cdot Db^*)\in F[y]$ in an extension field E of F, the $v_i\in E[x]$ are squarefree and pairwise coprime, and $b^*=v_1\cdots v_t$.)

If the *key equation*

$$r\cdot Du+s^*\cdot u=v, \tag{9}$$

where $s^*=s+Dr$, has a polynomial solution $u\in F[x]$, then $\tau=ur/v$ satisfies the Risch differential equation, and $f=\tau g$ is an integral of g. If, however, (9) is unsolvable, then g has provably no hyperexponential integral.

Given a degree bound on u, as provided by the following lemma (see, e.g., Bronstein [8], §6.3), the key equation can be solved by the method of undetermined coefficients.

Lemma 5 *Let $r,s^*,u,v\in F[x]$, with r monic, $\mathrm{lc}(s^*)=-\delta$ (where we set $\delta=0$ if $s^*=0$), and u,v nonzero, such that (9) holds.*

(i) $\deg u=\deg v-\max\{\deg s^*,\deg r-1\}$ *if* $\deg s^*\neq\deg r-1$,

(ii) $\deg u=1+\deg v-\deg r$ *if* $\deg s^*=\deg r-1$ *and* $\delta\notin\mathbb{N}$,

(iii) $\deg u\in\{1+\deg v-\deg r,\delta\}$ *if* $\deg s^*=\deg r-1$ *and* $\delta\in\mathbb{N}$.

(iv) *In the first two cases, (4) has at most one polynomial solution $u \in F[x]$. In the last case, we have $\delta \neq 1 + \deg v - \deg r$.*

Similarly as in the difference case, the key equation has at most one solution if

$$\frac{-s}{r} = \frac{Du^*}{u^*} \tag{10}$$

has no nonzero polynomial solution $u^* \in F[x]$, which is the case if and only if g is not rational.

The analog of (6) is to take $a = (b_1 - d)x + a_0$ and $b = x^2 + b_1x + b_0$ in $F[x]$, with $a_0, b_1, b_0 \in F$ and $d \in \mathbb{N}_{\geq 2}$, and $\sigma = a/b$.

Theorem 6 *If $a = (b_1 - d)x + a_0$ and $b = x^2 + b_1x + b_0$ in $F[x]$, with $a_0, b_1, b_0 \in F$ and $d \in \mathbb{N}_{\geq 2}$, then we have $v = 1$ in (8). If in addition there is no nonzero polynomial $u^* \in F[x]$ such that $Du^*/u^* = -a/b$, then (9) has a unique polynomial solution $u \in F[x]$ of degree $d - 2$.*

Again, the differential equation is unsolvable if d is an indeterminate.

Example 7 *Let $d \in \mathbb{N}_{\geq 3}$ be odd and $g = (x^2+1)^{-d/2}$. Then*

$$Dg = -\frac{d}{2}(x^2+1)^{-(d+2)/2} \cdot 2x = \frac{-dx}{x^2+1}g,$$

so that $b = x^2 + 1$ and $a = -dx$. Since d is odd, we have $\gcd(b, a - ib') = \gcd(x^2 + 1, -(d + 2i)x) = 1$ for all $i \in \mathbb{Z}$. Obviously g is not rational, and Theorem 6 implies that the differential equation (9) has a unique solution $u \in F[x]$ of degree $d - 2$ for $r = b = x^2 + 1$, $s^* = a + Db = -(d - 2)x$, and $v = 1$, and hence $f = urg = u \cdot (x^2+1)^{-(d-2)/2}$ satisfies $Df = g$.

For example, if $d = 11$ then

$$D\left(\frac{x(128x^8 + 576x^6 + 1008x^4 + 840x^2 + 315)}{315(x^2+1)^{9/2}}\right) = \frac{1}{(x^2+1)^{11/2}}.$$

We may write the integrand as $g = (x^2+1)^{-(d+1)/2}\sqrt{x^2+1}$, and the degree of the "rational part" $(x^2+1)^{-(d+1)/2}$ seems to determine the degree of u. A similar remark applies to Example 3, where we may split off a factorial power.

The example illustrates the the "frontier" between rational and proper hyperexponential problems is thin: for even d, we obtain a rational integrand, which has no rational integral if $d > 2$.

4 The q-difference case

In the q-hypergeometric case, almost everything goes through as in the hypergeometric case if we formally replace the shift-operator E by the *q-shift operator* $Q\rho = \rho(qx)$ on $F(x)$, where $q \in F$ is nonzero and not a root of unity (we may think of x being the function $k \mapsto q^k$ on $\mathbb{N}$, and then Q is E acting on the exponent). An element g in an extension field of $F(x)$ is *q-hypergeometric* if $Qg = \sigma g$ for a rational function $\sigma \in F(x)$. The *q-hypergeometric summation problem* is to find another q-hypergeometric expression f such that $\Delta_q f = Qf - f = g$. If such an f exists, then it is a rational multiple $f = \tau g$ of g, for some $\tau \in F(x)$. The equation $\Delta_q f = g$ is equivalent to the rational q-difference equation

$$\sigma \cdot Q\tau - \tau = 1. \tag{11}$$

We may compute the *q-Gosper-Petkovšek* representation

$$\sigma = \frac{r}{Qs} \cdot \frac{Qv}{v}, \quad \gcd(r, Q^i s) = 1 \text{ for } i \in \mathbb{N}_{>0}, \tag{12}$$

with polynomials $r, s, v \in F[x]$, which exists uniquely if we stipulate that s, v be monic and $\gcd(r, v) = \gcd(s, v) = 1$. A polynomial solution $u \in F[x]$ of the *key equation*

$$r \cdot Qu - s \cdot u = v \tag{13}$$

yields a solution $\tau = us/v$ of the rational equation (2), $f = \tau g$ satisfies $\Delta_q f = g$, and no such f exists if (2) is unsolvable. The following lemma provides a degree bound on u.

Lemma 8 *Let $r = r_m x^m + \cdots + r_0$, s, v, and u be nonzero polynomials in $F[x]$, with $r_m \in F \setminus \{0\}$ and s monic, such that (13) holds.*

(i) $\deg u = \deg v - \max\{\deg r, \deg s\}$ *if* $\deg r \neq \deg s$,

(ii) $\deg u = 1 + \deg v - \deg r$ *if* $\deg r = \deg s$ *and* $r_m^{-1} \neq q^\delta$ *for all* $\delta \in \mathbb{N}$,

(iii) $\deg u \in \{1 + \deg v - \deg r, \delta\}$ *if* $\deg r = \deg s$ *and* $r_m^{-1} = q^\delta$ *for some* $\delta \in \mathbb{N}$.

(iv) *In the first two cases, (13) has at most one polynomial solution $u \in F[x]$. In the last case, we have $\delta \neq 1 + \deg v - \deg r$.*

As in the difference case, the solution space of (13) is either one-dimensional or empty if g is rational. In the former case, there is one solution u of degree $1 + \deg v - \deg r$, and all other solutions are of the form $u + cu^*$ for some constant $c \in F$, where u^* is a nonzero solution of degree δ of the homogeneous equation $r \cdot Qu^* - s \cdot u^* = 0$, or equivalently,

$$\frac{s}{r} = \frac{Qu^*}{u^*}. \tag{14}$$

If g is *proper* q-hypergeometric (i.e., q-hypergeometric but not rational), then (14) is unsolvable and there is at most one solution in any case.

We let

$$\sigma = \frac{a}{b} = \frac{q^{-d}x + a_0}{x + b_0}, \tag{15}$$

with $a_0, b_0 \in F$ and $d \in \mathbb{N}_{\geq 1}$. Then a and b are *q-shift-coprime*, i.e., $\gcd(a, Q^i b) = 1$ for all $i \in \mathbb{Z}$, or equivalently, $\mathrm{res}_x(a(x), b(yx)) \in F[y]$ has no integer zeroes, unless $b_0 = a_0 q^i$ for some $i \in \mathbb{Z}$. Thus in contrast to the difference case, we need only linear polynomials.

Theorem 9 *Let $a = q^{-d}x + a_0$ and $b = x + b_0$ in $F[x]$, with $a_0, b_0 \in F$ such that $b_0 \neq a_0 q^i$ for all $i \in \mathbb{Z}$, and $d \in \mathbb{N}_{\geq 1}$. Then we have $v = 1$ in (12), and there is a unique polynomial $u \in F[x]$ of degree $d - 1$ solving (13).*

As in the ordinary difference case, the key equation has no solution if d is an indeterminate.

Example 10 *Let $d \in \mathbb{N}_{\geq 1}$, and define the 2-hypergeometric expression g by*

$$g(k) = \frac{(2^{-d}+1)(2^{-d+1}+1)\cdots(2^{-d+k-1}+1)}{2^{(k^2-k)/2}}$$

for all $k \in \mathbb{N}$. Then the term ratio is

$$\sigma(k) = \frac{g(k+1)}{g(k)} = \frac{2^{-d+k}+1}{2^k},$$

and we have $a = 2^{-d}x + 1$ and $b = x$. Theorem 9 implies that there is a unique polynomial $u \in F[x]$ of degree $d-1$ solving the key equation (13) for $r = a$, $s = Q^{-1}b = x/2$, and $v = 1$, and $f = u \cdot Q^{-1}b \cdot g$ satisfies $\Delta_q f = g$.

For example, if $d = 11$ then

$$\Delta\Bigg(2^k\bigg(\frac{10414855105976475}{1024}2^{10k} + \frac{10414855105976475}{512}2^{9k} + \frac{3471618368658825}{256}2^{8k} + \frac{495945481236975}{128}2^{7k} + \frac{33063032082465}{64}2^{6k} + \frac{1066549422015}{32}2^{5k} + \frac{16929355905}{16}2^{4k} + \frac{133302015}{8}2^{3k} + \frac{522753}{4}2^{2k} + \frac{1023}{2}2^k + 1\bigg)g(k)\Bigg) = g(k),$$

for all $k \in \mathbb{N}$.

5 Future work

It would be interesting to find explicit classes of proper (q)-hypergeometric (or hyperexponential) expressions such that the *denominator* v of τ has arbitrarily high degree, for small input degrees.

The remark at the end of Example 7 indicates that δ is related to the degree of the "rational part" of a hyperexponential integrand. It is desirable to make this statement more precise, and also its hypergeometric analog, and to cover both statements in a unified framework.

References

[1] Abramov, S. A. On the summation of rational functions. *Ž. Vyčisl. Mat. i Mat. Fiz. 11* (1971), 1071–1075. English translation in U.S.S.R. Computational Mathematics and Mathematical Physics.

[2] Abramov, S. A. Problems in computer algebra involved in the search for polynomial solutions of linear differential and difference equations. *Moscow Univ. Comput. Math. and Cybernet. 3* (1989), 63–68.

[3] Abramov, S. A. Rational solutions of linear differential and difference equations with polynomial coefficients. *U.S.S.R. Comput. Maths. Math. Phys. 29* (1989), 7–12.

[4] Abramov, S. A. Rational solutions of linear difference and q-difference equations with polynomial coefficients. In *Proc. 1995 Internat. Symp. Symb. Alg. Comput.* (1995), ACM Press, pp. 285–289.

[5] Abramov, S. A., Bronstein, M., and Petkovšek, M. On polynomial solutions of linear operator equations. In *Proc. 1995 Internat. Symp. Symb. Alg. Comput.* (1995), ACM Press, pp. 290–296.

[6] Bronstein, M. The transcendental Risch differential equation. *J. Symbolic Comput. 9* (1990), 49–60.

[7] Bronstein, M. The algebraic Risch differential equation. In *Proc. 1991 Internat. Symp. Symb. Alg. Comput.* (1991), ACM Press, pp. 241–246.

[8] Bronstein, M. *Symbolic Integration I – Transcendental Functions.* No. 1 in Algorithms and Computation in Mathematics. Springer-Verlag, Heidelberg, 1996.

[9] Davenport, J. H. The Risch differential equation problem. *SIAM J. Comput. 15* (1986), 903–918.

[10] Ekhad, S. B. A very short proof of Dixon's theorem. *J. Combin. Theory. Ser. A 54* (1990), 141–142.

[11] Ekhad, S. B., and Tre, S. A purely verification proof of the first Rogers-Ramanujan identity. *J. Combin. Theory. Ser. A 54* (1990), 309–311.

[12] von zur Gathen, J., and Gerhard, J. Fast algorithms for Taylor shifts and certain difference equations. In *Proc. 1997 Internat. Symp. Symb. Alg. Comput.* (Maui, 1997), ACM Press, pp. 40–47.

[13] Gosper, Jr., R. W. Decision procedure for indefinite hypergeometric summation. *Proc. Nat. Acad. Sci. U. S. A. 25* (1978), 40–42.

[14] Hermite, C. Sur l'intégration des fractions rationelles. *Nouvelles Annales de Mathématiques (2^{eme} série) 11* (1872), 145–148.

[15] Karr, M. Summation in finite terms. *J. Assoc. Comput. Mach. 28* (1981), 305–350.

[16] Karr, M. Theory of summation in finite terms. *J. Symbolic Comput. 1* (1985), 303–315.

[17] Lisoněk, P., Paule, P., and Strehl, V. Improvement of the degree setting in Gosper's algorithm. *J. Symbolic Comput. 16* (1993), 243–258.

[18] Moenck, R. On computing closed forms for summation. In *Proceedings of the 1977 MACSYMA Users Conference* (Berkeley, 1977), pp. 225–236.

[19] Paule, P. Greatest factorial factorization and symbolic summation. *J. Symbolic Comput. 20* (1995), 235–268.

[20] Paule, P., and Strehl, V. Symbolic summation — some recent developments. In *Computer Algebra in Science and Engineering* (Bielefeld, Germany, August 1994), World Scientific, pp. 138–162.

[21] Petkovšek, M. Hypergeometric solutions of linear recurrences with polynomial coefficients. *J. Symbolic Comput. 14* (1992), 243–264.

[22] Pirastu, R., and Strehl, V. Rational summation and Gosper-Petkovšek representation. *J. Symbolic Comput. 20* (1995), 617–635.

[23] Risch, R. H. The problem of integration in finite terms. *Trans. Amer. Math. Soc. 139* (1969), 167–189.

[24] Rothstein, M. *Aspects of symbolic integration and simplification of exponential and primitive functions.* PhD thesis, University of Wisconsin-Madison, 1976.

[25] Trager, B. M. Algebraic factoring and rational function integration. In *Proc. ACM Symp. Symbolic and Algebraic Computation* (ed. R. D. Jenks, Yorktown Heights, NY, 1976), pp. 219–226.

[26] Wilf, H. S., and Zeilberger, D. Rational functions certify combinatorial identities. *J. Amer. Math. Soc. 3* (1990), 147–158.

[27] Wilf, H. S., and Zeilberger, D. An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities. *Inventiones Math. 108* (1992), 575–633.

[28] Zeilberger, D. A fast algorithm for proving terminating hypergeometric identities. *Discrete Math. 80* (1990), 207–211.

[29] Zeilberger, D. A holonomic systems approach to special function identities. *J. Comput. Appl. Math. 32* (1990), 321–368.

[30] Zeilberger, D. The method of creative telescoping. *J. Symbolic Comput. 11* (1991), 195–204.